A General Principle for Solving Problems in Linear Algebra

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A large number of the "theoretical" (as opposed to computational) questions we've seen so far can be classified into two categories:

1. Questions about existence;
2. Questions about uniqueness.

The general principle is: to solve "existence" questions, row-reduce a matrix and look at the *rows*. To solve "uniqueness" questions, row-reduce a matrix and look at the *columns*.

That's all there is to it...well not quite; you have to decide: (i) Is my question about existence or uniqueness? (ii) Which matrix should I look at? (iii) How should I interpret the results?. But if you're stuck on a question and can't remember what to do, at least this idea will help you narrow the field.

Here are some examples of how some of the classic questions we've seen fit into this general framework:

- "Is this system consistent?", or "For which values of h (or b_1, b_2, b_3) is this system consistent?". Asking whether a system is consistent is asking whether there exists a solution. So this is an *existence* question. Row reduce the augmented matrix of the system, and look at the *rows*. The system is consistent iff there are no rows like $[0\ 0\ \ldots\ 0\ b]$, $b \neq 0$.

- "Given explicit vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$, and some other vector $\mathbf{u}$, does the vector $\mathbf{u}$ belong to $\mathrm{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$?" We are asking, does there exist a linear combination of the vectors $\mathbf{v}$ which equals $\mathbf{u}$? So this is another *existence* question. To answer it, row reduce the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_p\ \mathbf{u}]$ and look to see if there is a *row* like $[0\ 0\ \ldots\ 0\ b]$, $b \neq 0$. If there is such a row, $\mathbf{u}$ is not in the span of the $\mathbf{v}$'s. If there is no such row, then $\mathbf{u}$ is in the span of the $\mathbf{v}$'s.

- "Do the columns of the matrix A span $\mathbb{R}^m$?" This is asking, is it true that for every vector $\mathbf{b} \in R^m$ there exists a linear combination of the columns of A which is equal to $\mathbf{b}$? So this is also an *existence* question. To solve it, row-reduce A, and look at the *rows*. If there is a pivot in every row, the columns span $\mathbb{R}^m$. If there isn't, then they don't.

- "Is the linear transformation T onto?" This question is asking, is it true that for every $\mathbf{b}$ in the codomain of T there exists a vector $\mathbf{x}$ in the domain such that $T(\mathbf{x}) = \mathbf{b}$? So this again is an *existence* question. Row reduce the standard matrix of T, and look at the *rows*: if there's a pivot in every row, T is onto. If there's not, it's not.

- "Does the equation $A\mathbf{x} = \mathbf{0}$ have any nontrivial solutions?" We know that this equation has a solution (the trivial solution $\mathbf{x} = \mathbf{0}$). The question is, is this the only one? So this is a question of *uniqueness*. To answer it, row reduce the augmented matrix $[A\ \mathbf{0}]$ and look at the *columns*: i.e. look for free variables. If there's a free variable, the trivial solution is not unique, so there are nontrivial solutions. If there's no free variable, the trivial solution is unique.

- "Are these vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ linearly independent?" This is asking, are there numbers $x_1, \ldots, x_p$, not all zero, such that $x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$? This is again a *uniqueness* question: we know that the solution $x_1 = x_2 = \ldots = x_p = 0$ always satisfies this equation, the question is, is there another solution? So row-reduce the matrix $[\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_p]$ and look at the *columns*: if there's a pivot in every column, the vectors are independent. If there's not, they're not.

- "Is this linear transformation T one-to-one?" This is asking, is it possible that two different vectors have the same image under the transformation? So this too is a *uniqueness* question: if $T(\mathbf{x}) = \mathbf{b}$, is $\mathbf{x}$ the only vector that gets mapped to $\mathbf{b}$, or is there another one? So row-reduce the standard matrix of T and look at its *columns*. If there's a pivot in every column, T is one-to-one. If there's not, it's not.

The list could go on, but I hope you get the idea... if your question is about existence of some object, look at rows. If it's about uniqueness (aka existence of *at most one* object), look at columns. Maybe there's an exception to this general rule, but if there is I haven't been able to think of it.